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Stability Test for a Related Routh-Hurwitz Problem

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Introduction

THE necessary and sufficient conditions for the asymptotic stability of a system described by a set of linear ordinary differential equations with constant coefficients are given by the well-known Routh-Hurwitz criterion or by the existence of a quadratic Lyapunov function. Corresponding conditions for the stability (i.e., none of the characteristic roots have positive real parts, some may be distinct pure imaginaries) are sometimes of interest. It is true for practical systems that asymptotic stability is a much more important requirement than is stability. But the latter concept often arises in the idealized modeling of physical systems, for instance, in the rotation of a torque-free rigid body, and in the classical theory of the libration of the moon.

Necessary and sufficient conditions for stability may be obtained by an extension of the method used in the derivation of the Routh-Hurwitz criterion. Certain theorems in this connection are given in Lehnigk,[1] although the method is not well-documented. The purpose of the present article is to outline a step-by-step stability test procedure, utilizing simplifications provided by some recent results.

Consider a dynamic system with the following real characteristic equation

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0 \qquad (1)$$

If $a_n = 0$, Eq. (1) has at least a zero root, and can be reduced to a lower order system immediately. Therefore, assume $a_n = 0$ for convenience. Also define as special roots of Eq. (1) any roots $x = \pm x^*$.

It is useful to decompose $f(x)$ in the following two ways:

$$\begin{aligned} 1)\quad f(x) &= h(x^2) + xg(x^2) \\ h(x^2) &= a_n + a_{n-2}x^2 + \dots \qquad (2)\\ g(x^2) &= a_{n-1} + a_{n-3}x^2 + \dots \\ 2)\quad f(x) &= p(x)\,q(x) = p(x)s(x^2) \end{aligned}$$

where $p(x)$ is a polynomial without any special roots and $q(x) = s(x^2)$ is a polynomial with special roots only. If $f(x)$ has any pure imaginary roots, $q(x)$ will not be an identical constant. The decomposition Eq. (3) is the crux of the problem.

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Once this is accomplished, it follows the system; Eq. (1) is stable if and only if all roots of $p(x)$ have negative real parts and all roots of $q(x)$ are distinct pure imaginaries, or equivalently, $s(y)$ has only distinct negative roots. With these as preliminaries, a step-by-step stability test procedure is outlined in the following section.

A Step-by-Step Stability Test

1) If $f(x)$ has only special roots, i.e., if it takes on the form $s(x^2)$, go to step 5 below.

2) If $f(x)$ is not of the form $s(x^2)$, then if any of the coefficients a_i $(i = 1,2,\dots n)$ is negative or zero, the system is unstable, no need to proceed further.

3) Define, as usual, Hurwitz determinants $\Delta_1,\Delta_2,\dots\Delta_{n-1}$ as the successive principal minors of the following $(n \times n)$ Hurwitz matrix:

$$\begin{bmatrix} a_1 & a_3 & a_5 & \cdot & 0 & 0 \\ 1 & a_2 & a_4 & \cdot & 0 & 0 \\ 0 & a_1 & a_3 & \cdot & 0 & 0 \\ 0 & 1 & a_2 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdot & a_{n-2} & a_n \end{bmatrix}$$

If n is even, evaluate the Hurwitz determinants Δ_3, Δ_5, $\dots$ Δ_{n-1}. If n is odd, evaluate the Hurwitz determinants Δ_2, Δ_4, $\dots$ Δ_{n-1}. a) If any one of these Hurwitz determinants is negative, the system is unstable, no need to proceed further. b) If all Hurwitz determinants are positive, the system is asymptotically stable, no need to proceed further.

4) If the Hurwitz determinants satisfy the following condition,

$$\Delta_{n-1} = \Delta_{n-3} = \dots = \Delta_{n-2r+1} = 0, \qquad \Delta_{n-2r-1} \neq 0 \quad (4)$$

the system can still possible be stable, but not asymptotically stable. Now $p(x)$ must be a polynomial of degree $n - 2r$, and the Hurwitz determinants of $p(x)$ are identical to Δ_{n-m}, $m = 2r, 2r + 1, \dots$. It follows: a) if in addition to the conditions of Eq. (4), $\Delta_{n-2t-1} = 0$ for some $t > r$, then $p(x)$ has some root with a positive real part and the system is unstable, no need to proceed further. b) If in addition to conditions (4), $\Delta_{n-2t-1} > 0$ for all $t > r$, then $s(y = x^2) = y^r + b_1y^{r-1} + b_2y^{r-2} \dots + b_r$ is obtainable as follows. Replace the last element of the last row of the Hurwitz determinant Δ_{n-2r-1} by $g(y)$, the element above this by $h(y)$ the next element above by $yg(y)$, the next element above by $y^2g(y)$, etc. The resulting determinant is equal to $s(y)$, multiplied by a constant.

a) If any of the coefficients b_l $(l = 1,2\dots r)$ is negative or zero, $q(x)$ has some root with positive real part, the system is unstable, no need to proceed further. b) Form the polynomial

$$s(y^2) + ys'(y^2) = y^{2r} + b_1y^{2(r-1)} + b_2y^{2(r-2)} + \dots + b_r + ry^{2r-1} + b_1(r-1)y^{2r-3} + b_2(r-2)y^{2r-5} + \dots + b_{r-1}y$$

together with its associated $(2r + 2r)$ Hurwitz matrix

$$\begin{bmatrix} r & b_1(r-1) & b_2(r-2) & \cdot & 0 & 0 \\ 1 & b_1 & b_2 & \cdot & 0 & 0 \\ 0 & r & b_1(r-1) & \cdot & 0 & 0 \\ 0 & 1 & b_1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b_{r-1} & 0 \\ 0 & 0 & 0 & \cdot & b_{r-1} & b_r \end{bmatrix}$$

and evaluate the corresponding Hurwitz determinants. If all the Hurwitz determinants $\Delta_3 = b_1^2(r-1) - 2b_2r$, Δ_5, Δ_7, $\dots$ Δ_{2r-1} are positive $q(x)$ has only distinct pure imaginary roots and the system is stable. Otherwise the system is unstable. The above steps exhaust all possibilities. A brief discussion of these is given below.

Discussion

Test step 1 is trivial. The truth of test step 2 follows from the decomposition Eq. (3) and the application of the well-known condition for asymptotic stability to $p(x)$ and $s(x^2)$. Step 3 is well-known. Step 5a is again an application of the necessary condition for asymptotic stability to $s(y)$. Step 5b is an application of a theorem of Fuller.[2] The method to determine $s(y)$ in step 4 and the recognition that its Hurwitz determinants are identical to Δ_{n-m} are new and recent results.[3,4] The proofs lie in the identification of the Hurwitz determinants with the resultants and subresultants of the two subpolynomials $h(x^2)$ and $g(x^2)$ of the characteristic polynomial $f(x)$. Although this identification was made independently by the author,[3] a literature search subsequently revealed Fuller[2] seems to be the first author to make this identification by going back to the almost forgotten early work of Trudi. Householder[4] also discussed this identification in connection with the more general problem of a complex characteristic polynomial using the theorems of Trudi and Netto. Although Householder's work is not addressed directly to the present problem, his result is applicable if one converts the real polynomial $f(x)$ to a complex polynomial $w(z)$ by the transformation $x = iz$. Finally it should be pointed out the test steps outlined are essentially based on the Routh-Hurwitz algorith. It is conceivable other equivalent algorithms such as those suggested by Duffin[5] may be used to advantage in some of the intermediate steps.

Example

$$f(x) = x^9 + 3x^8 + \tfrac{9}{2}x^7 + \tfrac{23}{2}x^6 + 7x^5 + 14x^4 + \tfrac{9}{2}x^3 + \tfrac{13}{2}x^2 + x + 1$$

For this example,

$$h(x^2) = 3x^8 + \tfrac{23}{2}x^6 + 14x^4 + \tfrac{13}{2}x^2 + 1$$

$$g(x^2) = x^8 + \tfrac{9}{2}x^6 + 7x^4 + \tfrac{9}{2}x^2 \pm 1$$

$$\Delta_{n-1} = \Delta_8 = \begin{vmatrix} 3 & 23/2 & 14 & 13/2 & 1 & 0 & 0 & 0 \\ 1 & 9/2 & 7 & 9/2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 23/2 & 14 & 13/2 & 1 & 0 & 0 \\ 0 & 1 & 9/2 & 7 & 9/2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 23/2 & 14 & 13/2 & 1 & 0 \\ 0 & 0 & 1 & 9/2 & 7 & 9/2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 23/2 & 14 & 13/2 & 1 \\ 0 & 0 & 0 & 1 & 9/2 & 7 & 9/2 & 1 \end{vmatrix}$$

Straightforward computations give

$$\Delta_8 = \Delta_6 = \Delta_4 = 0,$$

$$\Delta_1 = 3 > 0,\ \Delta_2 = \begin{vmatrix} 3 & 23/2 \\ 1 & 9/2 \end{vmatrix} = 2 > 0,\ \Delta_3 = 2 > 0,$$

$$s(y = x^2) = 1/2 \begin{vmatrix} 3 & h(y) \\ 1 & g(y) \end{vmatrix} = y^3 + \tfrac{7}{2}y^2 + \tfrac{7}{2}y + 1 = x^6 + \tfrac{7}{2}x^4 + \tfrac{7}{2}x^2 + 1$$

An application of test step 5 shows the necessary and sufficient criteria for the stability of a system with the characteristic equation $X^6 + b_1X^4 + b_2X^2 + b_3 = 0$ are $b_1 > 0$, $b_2 > 0$, $b_3 > 0$, $b_1^2 - 3b_2 > 0$ and $b_2^2(b_1^2 - 4b_2) + b_1b_3(18b_2 - 4b_1^2) - 27b_3^2 > 0$. Obviously $s(x^2)$ given above satisfies these criteria and the system Eq. (5) is stable, although not asymptotically stable. It is now straightforward to show that indeed $f(x) = (x^3 + 3x^2 + x + 1)\cdot(x^6 + \tfrac{7}{2}x^4 + \tfrac{7}{2}x^2 + 1)$ and the Hurwitz determinants of $p(x) = x^3 + 3x^2 + x + 1$ agree with Δ_1, Δ_2, and Δ_3 given above.

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Mean Curvature of a Deformed Spherical Surface

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When studying the equilibrium configuration of liquid drops under the action of surface tension, it is necessary to compute the mean curvature, $H(\theta,\phi)$, at a given point on the surface. The difference in pressure across the surface of the drop, ΔP, is given by

$$\Delta p = \sigma H = \alpha/2(1/R_1 + 1/R_2)$$

where R_1 and R_2 are the principal radii of curvature at a point on the surface. An expression for $(1/R_1 + 1/R_2)$ is derived in Landau and Lifschitz[1] for the case of a surface given in spherical coordinates as $\eta(\theta,\phi)$. This expression, however, is a perturbation expansion about a sphere, good only to first order. In order to obtain higher order expansions we derive an exact analytical expression for the mean curvature using techniques of Differential Geometry.[2] Although useful in engineering applications, this result appears not to have been previously published.

Let the surface be given by the function $\eta(\theta,\phi)$ where θ,ϕ are the usual polar angles and η the distance from the origin. Then in Euclidean 3-space, the surface is represented by the vector function:

$$\mathbf{X}(\theta,\phi) = (\eta \sin\theta \cos\phi, \eta \sin\theta \sin\phi, \eta \cos\theta)$$

At a point on the surface there is a tangent plane spanned by the two vectors

$$\partial\mathbf{X}/\partial\theta \equiv \mathbf{X}_1 = ([\eta \cos\theta + \eta_\theta \sin\theta] \cos\phi, [\eta \cos\theta + \eta_\theta \sin\theta] \sin\phi, [\eta_\theta \cos\theta - \eta \sin\theta])$$

and

$$\partial\mathbf{X}/\partial\phi \equiv \mathbf{X}_2 = ([\eta_\theta \cos\phi - \eta \sin\phi] \sin\theta, [\eta_\phi \sin\phi + \eta \cos\phi] \sin\theta, [\eta_\phi \cos\theta])$$

A unit normal vector exists with respect to this plane. Since by convention a sphere has positive curvature the unit normal will be taken as pointing inwards, towards the center of the sphere. Thus the unit normal is:

$$\mathbf{X}_3 = -(\mathbf{X}_1 \times \mathbf{X}_2)/|\mathbf{X}_1 \times \mathbf{X}_2|$$

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